Zamolodchikov asymptotic formula and instanton expansion in $\mathcal{N} = 2$ SUSY $N_f = 2N_c$ QCD

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ABSTRACT: The AGT relations allow to convert the Zamolodchikov asymptotic formula for conformal block into the instanton expansion of the Seiberg-Witten prepotential for the theory with two colors and four fundamental flavors. This provides an explicit formula for the instanton corrections in this model. The answer is especially elegant for vanishing matter masses, then the bare charge of gauge theory $q_0 = e^{i\pi\tau_0}$ plays the role of a branch point on the spectral elliptic curve. The exact prepotential at this point is $\mathcal{F} = \frac{1}{2\pi i}a^2 \log q$ with $q \neq q_0$, unlike the case of another conformal theory, with massless adjoint field. Instead, $16q_0 = \theta_{10}^4/\theta_{00}^4(q) = 16q(1 + O(q))$, i.e. the Zamolodchikov asymptotic formula gives rise, in particular, to an *exact* non-perturbative beta-function so that the effective coupling differs from the bare charge by infinite number of instantonic corrections.

To the memory of Alesha Zamolodchikov

Contents

1 Introduction

The AGT relation [1]–[15], originally motivated by consideration of 5-brane compactifications on Riemann surfaces [16], expresses the $2d$ conformal blocks [17]–[22] through Nekrasov functions [23]–[31], which in the limit $\epsilon_1, \epsilon_2 \to 0$ reproduce [26, 27, 32] instanton contributions to the Seiberg-Witten prepotentials [33]–[53], describing the low-energy phases of multidimensional $\mathcal{N} = 2$ SUSY gauge theories. In the simplest case of the 4-point Virasoro conformal block, the relevant Seiberg-Witten model is $N_f = 2N_c$ gauge theory in $4d$ with $N_c = 2$, i.e. with two colors. This theory has vanishing beta-function, and the asymptotically free models with less flavors arise when some masses (and, therefore, some dimensions of external fields in the conformal blocks) get large and decouple [5, 13].

If instead one considers large intermediate dimension Δ, the situation gets even more interesting. When translated by the AGT relation into the language of SUSY gauge theory, the large Δ expansion turns into the instanton expansion (in $\Delta^{-1} \sim 1/a^2$). More accurately, the number of instantons is controlled by power of the double-ratio x-parameter in the conformal block, to be identified, up to a numeric factor, with the bare coupling $q_0 = e^{i\pi\tau_0} = e^{i\vartheta_0 - 8\pi^2/g_0^2}$. At the *conformal* point[1] with all vanishing masses of matter hypermultiplets, the prepotential is just $\mathcal{F} = \frac{1}{2}\tau a^2$ with no $1/a^2$-corrections, just as follows from dimensional reasoning. However, this does not mean that instanton corrections are absent! In fact, the effective coupling $\tau \neq \tau_0$ *differs* from the bare charge, which is an exact and explicit example of topological charge renormalization [54]–[57]. Finding an exact form for the

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[1] We emphasize that here "*conformal*" (in italic) refers to conformal invariance in $4d$, while the same word in "the conformal block" refers to $2d$ conformal invariance.

function $\tau = \tau(\tau_0)$ or, better, $q = q(q_0)$, is one of the long-standing problems in Seiberg-Witten theory (see e.g. [34, 36, 39, 40]).

Naively one may think that $\tau = \tau_0$, i.e. is not renormalized at all for vanishing masses in $N_f = 2N_c$ model, like it happens in the theory with massless adjoint supermultiplet [42, 43, 45, 51, 52]. The approach to Seiberg-Witten theory, based on spectral curves and integrable systems [35, 36, 38, 47] leaves this issue aside, since only the *effective* coupling τ is derived as the period matrix of the Seiberg-Witten curve. However, a direct instanton calculus (see e.g. [23, 29, 39]) immediately demonstrates that the dependence $\tau = \tau(\tau_0)$ on the bare charge τ_0 is nontrivial. This can be seen directly by extrapolating the Nekrasov functions from $N_f < 2N_c$ case to $N_f = 2N_c$: the result is obviously non-trivial (the instantonic contributions contain ratios $a_i^{N_f} / \prod_{j \neq i} a_{ij}$, clearly different from zero for non-vanishing values of the condensates). Still, this does not give immediately the dependence $q(q_0)$, which can be possibly extracted by a careful treatment of the quasiclassical regime along the lines of [26, 27, 32].

However, after discovery of the AGT relations, at least, for $N_c = 2$ and $N_f = 2N_c = 4$, one can simply *read* the answer for this dependence from the wonderful papers by Alesha Zamolodchikov [21], where the large-Δ asymptotics of the conformal block has been found. The leading behavior for the conformal block (exact at the *conformal point* and for $c = 1$) is just

$$\mathcal{B}_\Delta(x) \sim (16q)^\Delta \tag{1.1}$$

with[2]

$$x = 16q_0 = \frac{\theta_{10}^4(q)}{\theta_{00}^4(q)} = 16q \prod_{n=1}^\infty \left(\frac{1 + q^{2n}}{1 + q^{2n-1}} \right)^8 \tag{1.2}$$

where the Jacobi theta-constants are

$$\theta_{00}(q) = 1 + 2\sum_{n=1}^\infty q^{n^2} = \prod_{n=1}^\infty (1 - q^{2n})(1 + q^{2n-1})^2, \tag{1.3}$$

$$\theta_{10}(q) = 2q^{1/4} \sum_{n=0}^\infty q^{n(n+1)} = 2q^{1/4} \prod_{n=1}^\infty (1 - q^{2n})(1 + q^{2n})^2 \tag{1.4}$$

It means that the prepotential in the $N_f = 2N_c = 4$ Seiberg-Witten theory with vanishing masses is

$$\mathcal{F}(a) = \frac{1}{2\pi i} a^2 \log q = \frac{1}{2} a^2 \tau \tag{1.5}$$

which depends nontrivially over the bare coupling, in contrast to the classical prepotential

$$\mathcal{F}_{cl}(a) = \frac{1}{2\pi i} a^2 \log q_0 = \frac{1}{2} a^2 \tau_0 \tag{1.6}$$

due to (1.2). The classical prepotential (1.6) does not get corrections in perturbation theory, since the beta-function vanishes for $N_f = 2N_c$, but is nontrivially renormalized non-perturbatively.

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[2] Remarkably, this dressing formula reproduces the geometrical engineering prediction of ref. [1, 58]. We are indebted to Marcos Marino for attracting our attention to this important paper.

The spectral curve for the $N_c = 2$, $N_f = 4$ Seiberg-Witten theory can be written as [38, 47]

$$Y^2 = (X^2 - u)^2 - Q_4(X|\mu) \tag{1.7}$$

with the polynomial $Q_4(X|\mu; \tau)$ of degree 4 which depends on masses. It is endowed with the meromorphic generating differential

$$dS \sim x \left(\frac{2X dX}{Y} - \frac{X^2 - u}{2Y} \frac{dQ_4(X|\mu)}{Q_4(X|\mu)} \right) \tag{1.8}$$

with residues proportional to the fundamental masses. The prepotential is defined by $a^D = \frac{\partial \mathcal{F}}{\partial a}$, where

$$a = \oint_A dS, \quad a^D = \oint_B dS \tag{1.9}$$

At the *conformal* point this turns into

$$Y^2 = (X^2 - u)^2 - \zeta X^4 \tag{1.10}$$

$$dS \sim u \frac{dX}{Y} \tag{1.11}$$

so that the Seiberg-Witten form becomes *holomorphic*, while $\zeta = \zeta(\tau)$ is a dimensionless parameter, which can depend therefore only upon the effective coupling. One gets hence from (1.9), (1.11) that at the *conformal* point

$$a^D = \tau a \tag{1.12}$$

where τ is the modulus (period matrix) of (1.7) so that the prepotential is indeed given by (1.5).

The curve (1.11) can be re-written in the form

$$\eta^2 = \xi(\xi - 1)(\xi - x) \tag{1.13}$$

related to (1.11) by a fractional-linear transformation with[3]

$$\zeta = \frac{4x}{(1+x)^2} = \frac{4\theta_{00}^4 \theta_{10}^4}{(\theta_{00}^4 + \theta_{10}^4)^2} \tag{1.15}$$

Already the form of the elliptic curve (1.13) leads immediately to the relation (1.2). Indeed (see e.g. [60]), one can associate with this curve the following values of the Weierstrass $\wp$-function at half-periods: $e_1 = \frac{2-x}{3}$, $e_2 = \frac{2x-1}{3}$ and $e_3 = -\frac{1+x}{3}$. Then,

$$x = \frac{e_2 - e_3}{e_1 - e_3} = \kappa^2 = \frac{\theta_{10}^4(q)}{\theta_{00}^4(q)} = 16q \prod_{n=1}^{\infty} \left(\frac{1 + q^{2n}}{1 + q^{2n-1}} \right)^8 \tag{1.16}$$

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[3]Sometimes a different parametrization

$$\tilde{\zeta} = \frac{4x(x-1)}{(2x-1)^2} = -\frac{4\theta_{10}^4 \theta_{01}^4}{(\theta_{01}^4 - \theta_{10}^4)^2} \tag{1.14}$$

is chosen, which is related to (1.15) by modular transformation $\tilde{\zeta}(x) = \zeta(1 - x^{-1})$.

In the weak coupling limit, when $x \to 0$, $q \to 0$ and, therefore, $x = 16q_0$.
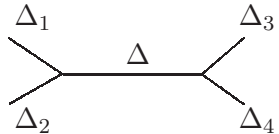
We see that in the parameterizations (1.13) the only nontrivial branching point coincides with the bare coupling (1.2)! The bare coupling therefore does not transform in a right way under the duality transformation and this is a geometric reason for the nontrivial instantonic renormalization (1.2). It deserves noting that the curve (1.13) has appeared in the study [21, 22] of the quasiclassical limit of conformal block.

Eq. (1.2) is the topological-charge renormalization formula of our main interest. At weak coupling the $q = e^{i\pi\tau}$-parameter of the torus is close to $q_0 = \frac{x}{16}$, but gets RG-dressed in this wonderful algebro-geometric way (quite similar to the *dream-like* behavior in other applications [56]).

Note, however, that extracting instanton expansion of the prepotential from the exact algebro-geometric solution is a nontrivial technical problem (see e.g. [27, 48, 50]). In the rest of the text we provide more technical details and illustrations about the large-Δ asymptotics of the AGT relation, using directly the Zamolodchikov asymptotic expansion. We use notation and formulas from [4].

2 Asymptotic formula for the conformal block [21]

By definition, the 4-point Virasoro conformal block is a formal series

$$
\mathcal{B}^{\Delta}_{\Delta_1\Delta_2;\Delta_3\Delta_4}(x) = \sum_{|Y|=|Y'|} x^{|Y|}\gamma_{\Delta\Delta_1\Delta_2}(Y)Q_{\Delta}^{-1}(Y,Y')\gamma_{\Delta\Delta_3\Delta_4}(Y') =
$$

$$
\tag{2.1}
$$

$$
= 1 + x\frac{D_{12}^{(1)}D_{34}^{(1)}}{2\Delta} + x^2\Big(D_{12}^{(2)}, \ D_{12}^{(1)}(D_{12}^{(1)}+1)\Big)\begin{pmatrix} \frac{c}{2}+4\Delta & 6\Delta \\ 6\Delta & 4\Delta(2\Delta+1)\end{pmatrix}^{-1}
$$

$$
\times \begin{pmatrix} D_{34}^{(2)} \\ D_{34}^{(1)}(D_{34}^{(1)}+1)\end{pmatrix} + \cdots
\tag{2.2}
$$

where we have introduced a condensed notation $D_{ij}^{(k)} = \Delta + k\Delta_i - \Delta_j$. In fact, this formal series is a nice analytic function of its arguments, possessing well-defined singularities and satisfying certain equations [17, 18, 21]. In particular, according to [21], the conformal block (2.2) can be also represented as

$$
\mathcal{B}^{\Delta}_{\Delta_1\Delta_2;\Delta_3\Delta_4}(x) = \left(\frac{16q}{x}\right)^{\Delta-\frac{c-1}{24}}(1-x)^{\frac{c-1}{24}-\Delta_1-\Delta_3}\theta_{00}(q)^{\frac{c-1}{2}-4(\Delta_1+\Delta_2+\Delta_3+\Delta_4)}H^{\Delta}_{\Delta_1\Delta_2;\Delta_3\Delta_4}(x),
\tag{2.3}
$$

where $H^{\Delta}_{\Delta_1\Delta_2;\Delta_3\Delta_4}(x) = 1 + \sum_{k>0} h_k q^k = 1 + O(\Delta^{-1})$ is equal to unity at $\Delta \to \infty$. This formula can be considered as a partial summation of the series (2.2), and q in (2.3) is related to x by eq. (1.2). The function $H^{\Delta}_{\Delta_1\Delta_2;\Delta_3\Delta_4}$ behaves nicely if considered as a function of q rather than x, still it is quite complicated and not well understood yet, for the latest results about this function see [59]. Here we only need its asymptotic behavior at large Δ.

In the rest of this section we present some illustration of the Zamolodchikov formula (2.3), demonstrating that it is indeed a very *explicit* statement. It has been proved long ago in [21], and we do not discuss the original proof in this letter.

2.1 Explicit expansions

Making use of (1.4), one can invert (1.2)

$$16q = x\left(1 + \frac{1}{2}x + \frac{21}{64}x^2 + \frac{31}{128}x^3 + \frac{6257}{32768}x^4 + \frac{10293}{65536}x^5 + \cdots\right) \tag{2.4}$$

and represent in a similar form the other entries of (2.3): the factor

$$
\begin{aligned}
\left(\frac{16q}{x}\right)^\Delta &= 1 + \frac{\Delta}{2}x + \left(\frac{21\Delta}{64} + \frac{\Delta(\Delta-1)}{8}\right)x^2 + \\
&\quad + \left(\frac{31\Delta}{128} + \frac{21\Delta(\Delta-1)}{128} + \frac{\Delta(\Delta-1)(\Delta-2)}{48}\right)x^3 + \\
&\quad + \left(\frac{6257\Delta}{32768} + \frac{1433\Delta(\Delta-1)}{8192} + \frac{21\Delta(\Delta-1)(\Delta-2)}{512}\right. \\
&\quad \left. + \frac{\Delta(\Delta-1)(\Delta-2)(\Delta-3)}{384}\right)x^4 + \cdots
\end{aligned}
\tag{2.5}
$$

and the theta-constant

$$\theta_{00}(q) = 1 + \frac{x}{8} + \frac{x^2}{16} + \frac{21x^3}{512} + \frac{993x^4}{32768} + \frac{6273x^5}{262144} + O(x^6) \tag{2.6}$$

For their logarithms one gets

$$\log\left(\frac{16q}{x}\right) = \frac{x}{2} + \frac{13x^2}{64} + \frac{23x^3}{192} + \frac{2701x^4}{32768} + \frac{5057x^5}{81920} + O(x^6) \tag{2.7}$$

$$\log\theta_{00}(q) = \frac{x}{8} + \frac{7x^2}{128} + \frac{13x^3}{384} + \frac{791x^4}{32768} + \frac{1523x^5}{81920} + O(x^6) \tag{2.8}$$

2.2 (2.2) vs (2.3): the first order in x

Making use of the first terms in (2.6) and (2.6), one gets from (2.3) in the first order in x:

$$
\begin{aligned}
1 + x &\left\{\frac{1}{2}\left(\Delta - \frac{c-1}{24}\right) + \left(\Delta_1 + \Delta_3 - \frac{c-1}{24}\right) + \right. \\
&\left. + \frac{1}{8}\left(\frac{c-1}{2} - 4(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4)\right)\right\}(1 + xh_1 + \cdots) =
\end{aligned}
\tag{2.9}
$$

$$= 1 + \frac{x}{2}\left(\Delta + \Delta_1 - \Delta_2 + \Delta_3 - \Delta_4 + 2h_1\right) + \cdots \tag{2.10}$$

Comparing this with the first-order term in (2.2), one obtains that

$$h_1 = \frac{(\Delta_1 - \Delta_2)(\Delta_3 - \Delta_4)}{2\Delta} \tag{2.11}$$

and, indeed, $h_1 = O(\Delta^{-1})$.

2.3 (2.2) vs (2.3): the second order in x at *conformal point*

In the second order in x formulas get more complicated, we begin with the simplified case, where all four external dimensions vanish $\Delta_{1,2,3,4} = 0$. Then, the series (2.2) becomes

$$
1 + x\frac{\Delta}{2} + x^2\Big(\Delta, \Delta(\Delta+1)\Big) \begin{pmatrix} \frac{c}{2} + 4\Delta & 6\Delta \\ 6\Delta & 4\Delta(2\Delta+1) \end{pmatrix}^{-1} \begin{pmatrix} \Delta \\ \Delta(\Delta+1) \end{pmatrix} + O(x^3) =
$$

$$
= 1 + \frac{x\Delta}{2} + \frac{x^2\Delta^2}{8} \cdot \frac{\Delta^2 + \left(4 + \frac{c}{2}\right)\Delta + (c-4) + \frac{c}{2\Delta}}{\Delta^2 + \frac{c-5}{8}\Delta + \frac{c}{16}} + O(x^3) =
$$

$$
= 1 + \frac{x\Delta}{2} + \frac{x^2}{8}\left(\Delta^2 + \frac{13\Delta}{8} + \frac{1-c}{64} + O(\Delta^{-1})\right) + O(x^3) \tag{2.12}
$$

Note that in these orders in x the c-dependence appears only in the $O(\Delta^{-1})$ terms. For vanishing external dimensions (2.3) reduces to

$$
\overbrace{\left(1 + \frac{\Delta-\sigma}{2}x + \left(\frac{13(\Delta-\sigma)}{64} + \frac{(\Delta-\sigma)^2}{8}\right)x^2 + \cdots\right)}^{(16q/x)^{\Delta-\sigma}}\overbrace{\left(1 - x\sigma + \frac{\sigma(\sigma-1)}{2}x^2 + \cdots\right)}^{(1-x)^\sigma} \cdot
$$

$$
\cdot \underbrace{\left(1 + \frac{12\sigma}{8}x + \left(\frac{12\sigma}{16} + \frac{12\sigma(12\sigma-1)}{2\cdot 8^2}\right)x^2 + \cdots\right)}_{\theta_{00}(q)^{12\sigma}}\left(1 + h_2 x^2 + \cdots\right) =
$$

$$
= 1 + \frac{x\Delta}{2} + \frac{x^2}{8}\left(\Delta^2 + \frac{13\Delta}{8} - \frac{3\sigma}{8} + h_2\right) + \cdots \tag{2.13}
$$

where $\sigma = \frac{c-1}{24}$ and we used the already evaluated (2.11) implying that $h_1 = 0$ at the *conformal point*. Comparing (2.13) with (2.12) one observes that, indeed, $h_2 = O(\Delta^{-1})$.

2.4 (2.2) vs (2.3): the second order in x

Restoring dependence on the external dimensions, one obtains instead of (2.12) and (2.13):

$$
1 + x\frac{(\Delta + \Delta_1 - \Delta_2)(\Delta + \Delta_3 - \Delta_4)}{2\Delta} +
$$

$$
+ x^2\Big(\Delta + 2\Delta_1 - \Delta_2, \ (\Delta + \Delta_1 - \Delta_2)(\Delta + \Delta_1 - \Delta_2 + 1)\Big) \cdot
$$

$$
\cdot \begin{pmatrix} \frac{c}{2} + 4\Delta & 6\Delta \\ 6\Delta & 4\Delta(2\Delta+1) \end{pmatrix}^{-1} \begin{pmatrix} \Delta + 2\Delta_3 - \Delta_4 \\ (\Delta + \Delta_3 - \Delta_4)(\Delta + \Delta_3 - \Delta_4 + 1) \end{pmatrix} + \cdots =
$$

$$
= 1 + x\frac{(\Delta + \Delta_1 - \Delta_2)(\Delta + \Delta_3 - \Delta_4)}{2\Delta} + \frac{x^2}{8}\left\{\Delta^2 + \left(\frac{13}{8} + 2(\Delta_1 - \Delta_2 + \Delta_3 - \Delta_4)\right)\Delta + \right.
$$

$$
+ \left(\frac{1-c}{64} + \frac{9(\Delta_1 + \Delta_3) - 7(\Delta_2 + \Delta_4)}{4} + (\Delta_1 - \Delta_2)^2 \right.
$$

$$
\left. + 4(\Delta_1 - \Delta_2)(\Delta_3 - \Delta_4) + (\Delta_3 - \Delta_4)^2\right) +
$$

$$+\left(\frac{(c+1)(c+5)}{512}-\frac{(c+3)(\Delta_1+\Delta_2+\Delta_3+\Delta_4)}{32}+\right.$$

$$+\frac{1}{8}\Big((\Delta_1-\Delta_2)^2+(\Delta_3-\Delta_4)^2+20(\Delta_1\Delta_3+\Delta_2\Delta_4)-12(\Delta_1\Delta_4+\Delta_2\Delta_3)\Big)+$$

$$\left.+16(\Delta_1-\Delta_2)(\Delta_3-\Delta_4)(\Delta_1-\Delta_2+\Delta_3-\Delta_4)\right)\frac{1}{\Delta}+O(\Delta^{-2})\Big\}+\cdots=$$

$$=\left\{1+\frac{x}{2}(\Delta+\Delta_1-\Delta_2+\Delta_3-\Delta_4)+\frac{x^2}{8}\left[\Delta^2+\left(\frac{13}{8}+2(\Delta_1-\Delta_2+\Delta_3-\Delta_4)\right)\Delta+\right.\right.$$

$$\left.\left.+\left(\frac{1-c}{64}+\frac{9(\Delta_1+\Delta_3)-7(\Delta_2+\Delta_4)}{4}+(\Delta_1-\Delta_2+\Delta_3-\Delta_4)^2\right)\right]+\cdots\right\}\cdot$$

$$\cdot\left\{1+x\frac{(\Delta_1-\Delta_2)(\Delta_3-\Delta_4)}{2\Delta}+\frac{x^2}{8\Delta}\left[\frac{(c+5)(c+1)}{512}-\frac{(c+3)(\Delta_1+\Delta_2+\Delta_3+\Delta_4)}{32}+\right.\right.$$

$$\left.+\frac{1}{8}\Big((\Delta_1-\Delta_2)^2+(\Delta_3-\Delta_4)^2+20(\Delta_1\Delta_3+\Delta_2\Delta_4)-12(\Delta_1\Delta_4+\Delta_2\Delta_3)\Big)\right.$$

$$\left.\left.+O(\Delta^{-1})\right]+\cdots\right\}\tag{2.14}$$

The l.h.s. of this equality represents the original expansion of the conformal block (2.2), while the r.h.s. represents the expansion of the Zamolodchikov formula (2.3). This ends our demonstrations of how eq. (2.3) works, which can be easily extended to higher orders with the help of computer simulations, and in the next section we are going directly to the consequences of the Zamolodchikov formula for the Seiberg-Witten prepotential.

3 Implication for the SW prepotential

The Nekrasov functions appear to be a clever regularization of the instantonic sums in multidimensional supersymmetric gauge theories, and originally auxiliary ϵ-parameters turn out to be the crucial modification. In particular, the AGT relation implies that the $2d$ central charge is $c=1+\frac{6(\epsilon_1+\epsilon_2)^2}{\epsilon_1\epsilon_2}$, so that the ϵ-parameters have a clear sense from the two-dimensional point of view. In order to get the Seiberg-Witten prepotentials, one should consider the limit, when both ϵ_1 and ϵ_2 go to zero, and extract the most singular term

$$\mathcal{F}_{\text{inst}}=\mathcal{F}-\mathcal{F}_{cl}=\lim_{\epsilon_1,\epsilon_2\to 0}\frac{-\epsilon_1\epsilon_2}{2\pi i}\log\left((1-x)^{-2\alpha_1\alpha_3/\epsilon_1\epsilon_2}\mathcal{B}^{\Delta}_{\Delta_1\Delta_2;\Delta_3\Delta_4}(x)\right)\tag{3.1}$$

where the parameter x is associated with the bare charge $x=16q_0=16e^{i\pi\tau_0}$ and the classical prepotential (1.6), while the external dimensions Δ_I are related to the fundamental masses μ_I, $I=1,\ldots,4$. If one puts $c=1$, the relation is especially simple: $\Delta_I=\frac{\alpha_I^2}{-\epsilon_1\epsilon_2}$ and

$$\mu_{1,2}=\alpha_1\pm\alpha_2,\qquad\mu_{3,4}=\alpha_3\pm\alpha_4\tag{3.2}$$

The same parameters α_1 and α_3 appear in the additional U(1)-factor $(1-x)^{-2\alpha_1\alpha_3/\epsilon_1\epsilon_2}$ at the r.h.s. of (3.1), this factor plays a crucial role in restoring the symmetry between

the four masses μ_I in the prepotential. The conformal block (2.2) is *not* symmetric in the external dimensions, since the vertex operators are located at different points on the Riemann sphere.

After that, the expansion of the asymptotic formula (2.3) considerably simplifies, since all the terms with $\sigma = \frac{c-1}{24}$ can be omitted, and only the leading powers of dimensions should be kept: the corrections vanish in the limit of small $\epsilon_{1,2}$-parameters. It is still not obvious how does the prepotential look like, and even why it is symmetric in $\mu_{1,2,3,4}$. Therefore, we start with an illustration, explicitly evaluating the first two orders in x.

3.1 Expansion of the prepotential from the conformal block

Taking logarithm of the r.h.s. of (2.14) one obtains:

$$\log \mathcal{B}(x) = \frac{x}{2}(\Delta + \Delta_1 - \Delta_2 + \Delta_3 - \Delta_4) + \frac{x^2}{8}\left(\frac{13\Delta}{8} + \frac{9(\Delta_1 + \Delta_3) - 7(\Delta_2 + \Delta_4)}{4}\right) + \cdots$$
$$\ldots + x\frac{(\Delta_1 - \Delta_2)(\Delta_3 - \Delta_4)}{2\Delta} +$$
$$+\frac{x^2}{64\Delta}\Big((\Delta_1 - \Delta_2)^2 + (\Delta_3 - \Delta_4)^2 + 20(\Delta_1\Delta_3 + \Delta_2\Delta_4) - 12(\Delta_1\Delta_4 + \Delta_2\Delta_3) +$$
$$+\frac{3}{16} - (\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4) + O(\Delta^{-1})\Big) + \cdots \tag{3.3}$$

The first line here comes from the logarithm of the Zamolodchikov asymptotics, i.e. from the divergent at $\Delta \to \infty$ part of (2.3); note that the Δ^2-terms growing as $(\epsilon_1\epsilon_2)^{-2}$ in the limit $\epsilon_{1,2} \to 0$ have disappeared after taking logarithm of (2.14). The other three lines in (3.3) present the contribution of the correction from $\log H = O(\Delta^{-1})$. The terms, which survive in this limit, are contained in the second and third lines, they are all of the form $\frac{\Delta_I\Delta_J}{\Delta}$, and the linear terms from the forth line disappear in the limit.

The first two lines in (3.3) are not symmetric in four masses $\mu_{1,2,3,4}$, which cannot be true for the prepotential: already in the linear order in x, according to (3.2), the term with $(\Delta_1 - \Delta_2)(\Delta_3 - \Delta_4) = \mu_1\mu_2\mu_3\mu_4$ is symmetric, while $\Delta_1 - \Delta_2 + \Delta_3 - \Delta_4 = \mu_1\mu_2 + \mu_3\mu_4$ is not. Restoring the symmetricity comes from the U(1)-factor in the r.h.s. of (3.1):

$$-2\alpha_1\alpha_3 \log(1 - x) = \frac{1}{2}(\mu_1 + \mu_2)(\mu_3 + \mu_4)\left(x - \frac{x^2}{2} + \cdots\right) \tag{3.4}$$

which converts the μ^2-term in the order x into the symmetric combination $\frac{x}{2}\sum_{i<j}\mu_i\mu_j$.

Analyzing in the same way the x^2-contributions, one gets

$$2\pi i \mathcal{F}_{\text{inst}} = a^2 \log\left(\frac{16q}{x}\right) + \left(\epsilon_1\epsilon_2(\Delta_1 + \Delta_3) - 2\alpha_1\alpha_3\right)\log(1 - x) +$$
$$+\epsilon_1\epsilon_2(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4)\log\left(\theta_{00}^4(q)\right) - \lim_{\epsilon_1\epsilon_2 \to 0}\epsilon_1\epsilon_2\left(xh_1 + x^2\left(h_2 - \frac{h_1^2}{2}\right) + \cdots\right) =$$

$$
= x\left(\frac{a^2}{2}+\alpha_1^2+\alpha_3^2+2\alpha_1\alpha_3-\frac{1}{2}(\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2)-\frac{(\alpha_1^2-\alpha_2^2)(\alpha_3^2-\alpha_4^2)}{2a^2}\right)+
$$

$$
+\frac{x^2}{8}\left(\frac{13a^2}{8}+\frac{1}{4}(9\alpha_1^2-7\alpha_2^2+9\alpha_3^2-7\alpha_4^2+32\alpha_1\alpha_3)-\right.
$$

$$
\left.-\frac{(\alpha_1^2-\alpha_2^2)^2+(\alpha_3^2-\alpha_4^2)^2+20(\alpha_1^2\alpha_3^2+\alpha_2^2\alpha_4^2)-12(\alpha_1^2\alpha_4^2+\alpha_2^2\alpha_3^2)}{8a^2}+O(a^{-4})\right)
$$

$$
+O(x^3)=
$$

$$
= \frac{x}{2}\left(a^2+\underbrace{\mu_1\mu_2}_{\alpha_1^2-\alpha_2^2}+\underbrace{\mu_3\mu_4}_{\alpha_3^2-\alpha_4^2}+\underbrace{(\mu_1+\mu_2)}_{2\alpha_1}\underbrace{(\mu_3+\mu_4)}_{2\alpha_3}-\frac{\mu_1\mu_2\mu_3\mu_4}{2a^2}\right)+
$$

$$
+\frac{x^2}{8}\left(\frac{13a^2}{8}+\frac{9(\mu_1+\mu_2)^2-7(\mu_1-\mu_2)^2+9(\mu_3+\mu_4)^2}{16}\right.
$$

$$
+\frac{-7(\mu_3-\mu_4)^2+32(\mu_1+\mu_2)(\mu_3+\mu_4)}{16}-
$$

$$
-\frac{\mu_1^2\mu_2^2+\mu_3^2\mu_4^2+\frac{5}{4}\left((\mu_1+\mu_2)^2(\mu_3+\mu_4)^2+(\mu_1-\mu_2)^2(\mu_3-\mu_4)^2\right)}{8a^2}+
$$

$$
\left.+\frac{\frac{3}{4}\left((\mu_1+\mu_2)^2(\mu_3-\mu_4)^2+(\mu_1-\mu_2)^2(\mu_3+\mu_4)^2\right)}{8a^2}+O(a^{-4})\right)+O(x^3)=
$$

$$
= \frac{x}{2}\left(a^2+\sum_{i<j}\mu_i\mu_j-\frac{\mu_1\mu_2\mu_3\mu_4}{2a^2}\right)+
$$

$$
+\frac{x^2}{64}\left(13a^2+\sum_i\mu_i^2+16\sum_{i<j}\mu_i\mu_j-\frac{1}{a^2}\sum_{i<j}\mu_i^2\mu_j^2-\frac{16\mu_1\mu_2\mu_3\mu_4}{a^2}+O(a^{-4})\right)
$$

$$
+O(x^3) \tag{3.5}
$$

One has obtained, therefore, the instanton expansion of the prepotential in the SU(2) $\mathcal{N}=2$ supersymmetric gauge theory with $N_f=4$ matter hypermultiples directly by expansion of the Zamolodchikov asymptotic formula (2.3). We see now that the coefficients in front of different structures in (3.5) are dressed *differently* by the instanton corrections. Using the algorithm of the previous section, such a calculation can be easily continued in order to get higher-instanton corrections to the Seiberg-Witten prepotential.

3.2 Exact renormalization from Zamolodchikov formula

The Zamolodchikov formula allows one also to find renormalization of three structures *exactly* in all orders of x:

$$
2\pi i\mathcal{F}_{\rm inst} = a^2\log\left(\frac{16q}{x}\right)-\frac{1}{4}\left(\sum_{i=1}^4\mu_i\right)^2\log(1-x)-\frac{1}{2}\sum_{i=1}^4\mu_i^2\log\left(\theta_{00}^4(q)\right)+O(a^{-2}) \tag{3.6}
$$

just coming from the divergent at $\Delta\to\infty$ part of (2.3). Two of the structures are independent of a and, therefore, are not well controlled by the spectral curve and formulas (1.9).

More important, one should remember that, from the point of view of Seiberg-Witten theory, the AGT relation and, therefore, the results (3.6) and (3.5) describe the instanton corrections computed by the Nekrasov algorithm [23], i.e. computed using the ϵ-regularized moduli space of instantons. In the case of $N_f = 2N_c$ they are different from the corrections, evaluated earlier with the help of singular moduli space (see pp. 190-204 of [53] for discussion of this difference and references). For example, (3.5) contains the contribution of one instanton, and generally the sectors with odd instantonic charges give nontrivial terms into $\mathcal{F}_{\text{inst}}$. Also, the two-instanton contribution in (3.5) is different from $\frac{7x^2a^2}{64\cdot243}$ found in [39].

It would be interesting to find a polynomial $Q_4(X|\mu;\tau)$ in (1.7) for non-vanishing masses $\mu_{1,2,3,4}$, which reproduces the answer (3.5).

4 Conclusion

To conclude, in this letter we have studied the large-Δ asymptotics of the AGT relation for the 4-point conformal block. This adds only a little to testing the AGT relations themselves, but this is hardly the main point of interest now, since a lot of evidence has been already collected. More important, we proposed an application to the beautiful asymptotic formula (2.3), found by Alesha Zamolodchikov many years ago [21]. The AGT relation converts this formula into the asymptotics of the Nekrasov partition function, which turns itself into the Seiberg-Witten prepotential for still mysterious 4d *conformal* theory with $N_f = 2N_c$. The Zamolodchikov formula is valid for the Virasoro conformal block, transferred by the AGT relation into the simplest case of the non-Abelian $\mathcal{N} = 2$ supersymmetric gauge theory with $N_c = 2$, nevertheless the result is spectacular. It confirms the expectation that the instanton corrections are non-trivial, despite the theory is conformal invariant, thus providing an example of purely non-perturbative renormalization group in supersymmetric gauge theory (not only in the sector of the topological charge, but also of the ordinary coupling constant). Moreover, the renormalization group evolution is described here by the nice algebro-geometric formula (1.2). Further understanding of this result and its generalization to $N_c > 2$ would be very interesting from many points of view.

Acknowledgments

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